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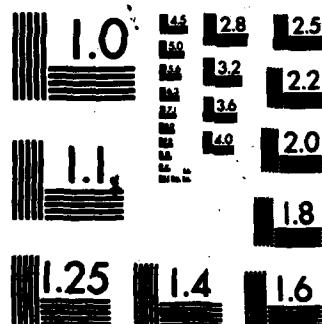
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FORMATION OF SINGULARITIES FOR A  
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FORMATION OF SINGULARITIES FOR A CONSERVATION LAW WITH MEMORY

Raza Malek-Madani\* and John A. Nohel

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ABSTRACT

The formation of singularities in smooth solutions of the model Cauchy problem

$$u_t + \phi(u)_x + a * \psi(u)_x = 0, \quad x \in \mathbb{R}, \quad t \in [0, \infty)$$

$$u(x, 0) = u_0(x)$$

is studied. The constitutive functions  $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$  are smooth,  $a : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a given memory kernel, subscripts denote partial derivatives,  $' = d/dt$  and  $*$  denotes the convolution on  $[0, t]$ . Under physically reasonable assumptions concerning the functions  $\phi, \psi$  and  $a$  it is shown that a smooth solution  $u$  develops a singularity in finite time, whenever the smooth datum  $u_0$  becomes "sufficiently large" in a precise sense.

AMS (MOS) Subject Classifications: 35L65, 35L67, 45G10, 45K05, 45D05, 45M10, 45M99, 47H10, 47H15, 47H17, 73P15, 73H10, 76N15

Key Words: conservation laws, Burger's equation, nonlinear viscoelastic motion, materials with memory, stress-strain relaxation functions, nonlinear Volterra equations, hyperbolic equations, dissipation, global smooth solutions, energy methods, asymptotic behaviour, method of characteristics, Riemann invariants, regularity, breakdown of smooth solutions

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## SIGNIFICANCE AND EXPLANATION

Problems arising in continuum mechanics can often be modeled by quasilinear hyperbolic systems in which the characteristic speeds are not constant. Such systems have the property that waves may be amplified and solutions that were initially smooth may develop discontinuities ("shocks") in finite time. Of particular interest are situations in which the destabilizing mechanism arising from nonlinear effects can coexist and compete with dissipative effects. An interesting situation arises when the amplification and dissipative mechanisms are nearly balanced and the outcome of their confrontation cannot be predicted at the outset. Examples are provided by quasilinear second order wave equations with first order frictional damping; it has been shown that when the initial data are sufficiently smooth and "small" in suitable norms, classical solutions exist globally in time. However, if the smooth initial data become sufficiently "large" in a precise sense, the smooth solution develops a singularity in finite time, no matter how smooth one takes the data. Thus the dissipative mechanism is not sufficiently powerful to prevent the breaking of waves for large enough data.

A considerably subtler dissipative mechanism is induced by memory effects of elastico-viscous materials. Using energy methods Dafermos and Nohel [1] have studied the motion of a one-dimensional homogeneous viscoelastic body (governed by equations (1.2), (1.3) below). They show that the memory term in (1.3) induces a weak dissipative mechanism under physically reasonable constitutive assumptions, which, for sufficiently "small" and smooth initial displacements and velocities, prevents the breaking of waves; indeed, a unique classical solution exists globally in time, and the solution decays as  $t \rightarrow \infty$ . A natural and open question (except in very special cases) is whether this weak dissipative mechanism can also prevent the breaking of waves for large enough smooth initial data; experimental evidence suggests that it cannot.

In order to gain a deeper understanding of this complex phenomenon we study the simpler model problem stated in the abstract, under comparable constitutive assumptions concerning the functions  $\phi$ ,  $\psi$  and  $a$ . Here the weak dissipative mechanism which is induced by the memory term acts exactly as the one for the viscoelastic problem for sufficiently smooth and small data (cf. [8]). In this paper we show, under physically reasonable constitutive assumptions, that this weak dissipative mechanism cannot overcome the shock forming tendency of the nonlinear Burgers operator  $\frac{\partial u}{\partial t} + \phi(u)$  when  $\phi$  is convex; indeed, a singularity develops in the smooth solution in finite time, whenever the smooth initial datum  $u_0(x)$  has  $u'_0(x) < 0$  and  $(-u'_0(x))$  is sufficiently large.

It is possible to gain some insight into the problem under study by considering the following simple example without memory terms:

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

$$u_t + uu_x + cu = 0$$

$$u(x, 0) = u_0(x) \quad (x \in \mathbb{R}) ;$$

in (0.1) the memory term is replaced by  $cu$  where  $c > 0$  is a constant, and  $\phi(u) = u^2/2$ , is a strictly convex function on  $\mathbb{R}$ . If  $u_0$  is smooth ( $C_0^\infty(\mathbb{R})$ ), (0.1) has a unique classical solution  $u$  valid on a maximal interval  $\mathbb{R} \times [0, T_0)$ ,  $0 < T_0 < \infty$ . Suppose the solution  $u$  of (0.1) exists globally in  $t$ . Differentiate (0.1) with respect to  $x$  obtaining

$$u_{tx} + uu_{xx} + u_x^2 + cu_x = 0 .$$

Putting  $w = u_x$  and noting that  $w_t + uw_x$  is the derivative of  $w$  along the characteristic curves  $x(t, \xi)$  of (0.1) defined by the ODE

$$\frac{dx}{dt} = u(x(t, \xi), t), \quad x(0, \xi) = \xi ,$$

we see that  $w$  satisfies the ODE

$$(0.2) \quad \frac{dw}{dt} + w^2 + cw = 0, \quad w(x, 0) = u'_0(x) ,$$

along the characteristics, where  $\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}$ . Integration of (0.2) shows that if  $u'_0(x) > -c$  ( $x \in \mathbb{R}$ ),  $w = u_x$  remains bounded for all  $t > 0$  and the smooth solution  $u$  of (0.1) exists globally; if, however,  $u'_0(x) < -c$  for

some  $x$ , then  $w = u_x \rightarrow -\infty$  as  $t \rightarrow \frac{1}{c} \log \frac{u'_0(x)}{c + u'_0(x)}$ , i.e., the classical

solution  $u$  of (0.1) develops a singularity in the first derivatives in finite time, no matter how smooth the initial datum  $u_0$  is taken. This elementary method does not, unfortunately, extend to the problem with memory terms under study, and for this reason our analysis is different and necessarily considerably more technical.

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# FORMATION OF SINGULARITIES FOR A CONSERVATION LAW WITH MEMORY

Rasa Malek-Madani\* and John A. Nohel

## I. INTRODUCTION.

In this paper we study the model initial value problem

$$\begin{aligned} u_t + \phi(u)_x + a * \phi(u)_x &= 0, \quad x \in \mathbb{R}, \quad t \in [0, \infty) \\ u(x, 0) &= u_0(x), \end{aligned} \quad (1.1)$$

where  $\phi, \phi : \mathbb{R} \rightarrow \mathbb{R}$  are given smooth constitutive functions,  $a : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a given kernel, subscripts denote partial derivative,  $' = d/dt$ , and where  $*$  denotes the usual convolution operator

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau.$$

The goal is to investigate the formation of singularities in finite time of classical solutions of (1.1) when the datum  $u_0$  is smooth. The motivation for studying equation (1.1) is provided by the more complex problem of the motion of a one-dimensional homogeneous viscoelastic body governed by the equation

$$u_{tt} - \sigma_x = 0, \quad (1.2)$$

together with appropriate initial and homogeneous boundary conditions; in (1.2) the stress  $\sigma$  is related to the strain  $u_x$  by the constitutive relation

$$\sigma(u_x) = \phi(u_x) + \int_0^t a'(t - \tau)\phi(u_x(x, \tau))d\tau. \quad (1.3)$$

Under appropriate physical assumptions concerning  $\phi, \phi$  and  $a$ , the "memory" term in (1.3) induces a weak dissipation mechanism into the structure of the solutions of (1.2). It has been shown (cf. Dafermos and Nohel [1]) that under physically proper assumptions on

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$a, \phi, \psi$  and on the initial data  $u_0$  and  $u_1$ , the initial-boundary value problem (1.2) has a unique global  $C^2$  solution, if the initial data are sufficiently smooth and "small" in an appropriate sense; moreover, this solution decays in a precise sense as  $t \rightarrow \infty$ . A similar behavior is exhibited by the solution  $u$  of (1.1) with  $u$  satisfying periodic boundary conditions (cf. Mohel [8]). These two results are of special interest since when  $a'(t) \equiv 0$ , (1.1) reduces to the Burgers equations, while (1.2), (1.3) reduce to the quasilinear wave equation  $u_{tt} = u(u_x)_x$ . For these problems it is well known (cf. Lax [5]) that under appropriate convexity assumptions on  $\phi$  there are smooth solutions which develop a singularity in the highest derivatives in finite time, no matter how smooth and small one chooses the initial datum. Thus  $a'(t) \not\equiv 0$  induces a weak dissipation mechanism which prohibits the breaking of waves when the initial amplitude of these waves is small.

This paper considers the natural question of how "large" one must choose the smooth initial datum in order that the shock forming structure of (1.1) overcomes this dissipation. Indeed, in Theorem 2.3 we show, under natural assumptions concerning the constitutive functions  $\phi, \psi$ , the kernel  $a$ , and datum  $u_0$ , that the classical solution  $u$  of (1.1) develops a singularity in  $u_x$  (and hence also in  $u_t$ ) in finite time for smooth and sufficiently "large" datum  $u_0$ . Our ultimate objective is to prove such a result for the complicated problem (1.2), (1.3), and with  $\phi \not\equiv \psi$ .

Equation (1.1) has a simpler structure than (1.2) due to the fact that (1.1) has only one family of "genuinely nonlinear" characteristics and one "linearly degenerate" characteristic due to the convolution term. Our approach examines the variation of the solution of (1.1) along characteristics with the aid of Riemann invariants. A similar approach (under active study) appears promising for the more complicated higher order problem (1.2), (1.3); this latter equation has three families of characteristics (only two are genuinely nonlinear), and thus, in general (1.2), (1.3) does not have Riemann invariants. Introducing the generalized Riemann invariants (cf. John [4]) there is reason to expect that much of our analysis can be adapted for (1.2), (1.3).

Some experimental evidence for the breakdown of smooth solutions of model equations governing viscoelastic materials can be found in the work of Tordella [10]. In addition



some results on the loss of regularity in solutions of the equations governing viscoelastic fluids, for smooth and sufficiently large data, have been obtained by Slemrod [9], and for dissipative hyperbolic volterra problems by Gripenberg [2] and Matteri [3] for the special cases of (1.2), (1.3) when  $\psi \equiv \phi$ . By methods similar to ours in spirit they also analyze the behavior of solutions along characteristics; however, they do not study the generalization to the more natural and more difficult situation in which  $\psi \neq \phi$ .

In Section 2 we state and discuss our assumptions and the main result; its proof is presented in Section 3. In Section 4 we prove two auxiliary results in the proof. We thank our colleagues, particularly C. M. Dafermos, R. Glassey, J. U. Kim, and M. Slemrod for helpful discussions.

## 2. ASSUMPTIONS AND STATEMENTS OF RESULT.

The basic constitutive assumption concerning  $\phi$  is

$$\phi \in C^2(\mathbb{R}) \text{ and } \phi'(\cdot) > 0, \phi''(\cdot) > 0, \phi(0) = 0. \quad (2.1)$$

The constitutive assumption concerning  $\psi$  is

$$\psi \in C^2(\mathbb{R}) \text{ and } \psi'(\cdot) > 0, \psi(0) = 0. \quad (2.2)$$

In addition, we assume that  $\phi$  and  $\psi$  are related as follows. There exists a constant  $\beta > 0$  such that

$$0 < \psi'(u) < \beta \phi'(u), \quad u \in \mathbb{R}. \quad (2.3)$$

Obviously, (2.3) is more restrictive than the assumption  $\phi'(0) > a(0)\psi'(0)$  (i.e. (2.3) at  $u = 0$  with  $\beta = a(0)^{-1}$ ) which was sufficient for the analysis of global solutions of (1.2), (1.3) in [1] for smooth and sufficiently small data. Assumption (2.3) simplifies our relatively technical analysis of the development of singularities for solutions of (1.1); in Remark 2.5 below we point out how (2.3) can be relaxed. Concerning the memory kernel  $a$  we assume that it is positive, decreasing and convex in the sense

$$a \in C^2[0, \infty), \quad (-1)^i a^{(i)}(t) > 0 \quad (i = 0, 1, 2), \quad (2.4)$$

where the strict inequalities hold at  $t = 0$ . Finally, we assume that the datum  $u_0$  satisfies

$$u_0 \in H^2(\mathbb{R}) ; \quad (2.5)$$

observe that  $u_0 \in H^2(\mathbb{R})$  implies  $u_0 \in C^1(\mathbb{R})$ .

Under assumptions which include (2.1), (2.2), (2.4), (2.5) as special cases the Cauchy problem (1.1) has a unique classical local solution. For this argument (2.3) is not used. More precisely, the following local result, proved by an energy method coupled with a contraction mapping argument, holds (cf. Mohel [8]).

Proposition 2.1. Let  $a', a'' \in (0, \infty)$ ,  $\phi, \psi \in C^2(\mathbb{R})$ ,  $\phi(0) = \psi(0) = 0$ ,  $\phi'(\cdot) > 0$ , and let there exist a constant  $\kappa$  such that  $\phi'(\xi) > \kappa > 0$  ( $\xi \in \mathbb{R}$ ). If  $u_0 \in H^2(\mathbb{R})$ , there exists  $T > 0$  and a unique solution  $u \in C^1(\mathbb{R} \times [0, T])$  of (1.1) such that  $u_{tt}, u_{tx}, u_{xx} \in C([0, T]; L^2(\mathbb{R}))$ .

Remark 2.2. It is also shown in [8] that the unique solution  $u$  exists on a maximal interval  $[0, T_0) \times \mathbb{R}$ ; if  $T_0 < \infty$ , then

$$\lim_{t \rightarrow T_0} \sup_{\mathbb{R}} \int [u^2(x, t) + u_t^2(x, t) + u_x^2(x, t) + u_{tt}^2(x, t) + u_{tx}^2(x, t) + u_{xx}^2(x, t)] dx = \infty.$$

Our main result is

Theorem 2.3. Let the assumptions (2.1)-(2.5) be satisfied, and let  $T_1 > 0$  be given. There exists smooth initial datum  $u_0$  such that no  $C^1$ -smooth solution  $u$  of (1.1) can exist for  $x \in \mathbb{R}$  and  $t > T_1$ . More precisely, if  $\sup_{x \in \mathbb{R}} |u_0(x)|$  is sufficiently small, and  $u_0'(x) < 0$  with  $-\inf_{x \in \mathbb{R}} u_0'(x)$  is sufficiently large, then the function  $u_x(x, t)$  (and hence also  $u_t(x, t)$ ) becomes negatively infinite for some  $t_1 < T_1$ , provided the smooth solution  $u$  exists on  $[0, t_1) \times \mathbb{R}$ .

Remark 2.4. While Theorem 2.3 establishes breakdown of smooth solutions of (1.1) for sufficiently large data, it does not prove the development of a shock front. Numerical evidence for this more complex phenomenon has been found by Markowich and Renardy [7] for the Cauchy problem associated with (1.2), (1.3) in the special case  $\phi \equiv \psi$  when the smooth data are taken sufficiently large. The corresponding analytical problem is under active study.

Remark 2.5. Theorem 2.3 holds if assumption (2.3) is satisfied only at  $u = 0$ . For, in this case there exists a constant  $B > 0$  such that (2.3) holds on the interval  $|u| < B$ , and the analysis of Section 3 can be modified accordingly.

Remark 2.6. It is also clear from the proof (cf. proof of Lemma 3.2) if the assumption  $u'_0(x_0) < 0$  and  $-u'_0(x_0)$  sufficiently large holds at a single point  $x_0$ .

### 3. PROOF OF THEOREM 2.3.

The proof is by contradiction. Assume that for any  $T_1 > 0$  and for every datum  $u_0$  satisfying (2.5) the unique smooth solution  $u$  of (1.1) exists for  $(x, t) \in \mathbb{R} \times [0, T_1]$  and that  $u_x(x, t)$  and  $u_t(x, t)$  are bounded on  $\mathbb{R} \times [0, T_1]$ . We begin by transforming (1.1) to an equivalent system. Let  $u$  be a smooth solution of (1.1) on  $\mathbb{R} \times [0, T_1]$  and introduce the dependent variable  $z$  by

$$z(x, t) = \int_0^t a'(t - \tau) \phi(u(x, \tau)) d\tau, \quad (x, t) \in \mathbb{R} \times [0, T_1]. \quad (3.1)$$

Equation (1.1) is then equivalent to the system

$$\begin{aligned} u_t + \phi(u)_x + z_x &= 0 \\ z_t &= a'(0)\phi(u) + a''\phi \end{aligned} \quad (x, t) \in \mathbb{R} \times [0, T_1], \quad (3.2)$$

together with the initial data  $u(x, 0) = u_0(x)$ ,  $z(x, 0) = 0$ . We next introduce

$\underline{u} = [u, z]^T$  and the matrices

$$A(\underline{u}) = \begin{bmatrix} \phi'(u) & 1 \\ 0 & 0 \end{bmatrix}, \quad B(\underline{u}, t) = \begin{bmatrix} 0 \\ -a'(0)\phi(u) - a''\phi \end{bmatrix},$$

then (3.2) can be written as the equivalent quasilinear system

$$\underline{u}_t + A(\underline{u})\underline{u}_x + B(\underline{u}, t) = 0, \quad \underline{u}(x, 0) = [u_0(x), 0]^T.$$

The  $2 \times 2$  matrix  $A(\underline{u})$  has distinct eigenvalues  $\phi'(u) > 0$  and  $0$ . A well known theorem of Lax [8] guarantees the existence of two linearly independent Riemann invariants  $r(u, z)$  and  $s(u, z)$ . By definition  $r$  and  $s$  satisfy

$$\begin{aligned} \xi_1 \cdot \nabla r &= 0 \\ \xi_2 \cdot \nabla s &= 0 \end{aligned} \quad (3.3)$$

where  $\xi_1$  and  $\xi_2$  are the right eigenvectors of  $\Lambda(U)$ . A simple calculation shows that  $\xi_1 = [1, -\phi'(u)]^T$  and  $\xi_2 = [1, 0]^T$ . It is then easy to show that

$$\begin{aligned} r(u, z) &= z + \phi(u) \\ s(u, z) &= z. \end{aligned} \quad (3.4)$$

satisfy (3.3), and moreover, by assumption (2.1),  $\frac{\partial(r, s)}{\partial(u, z)} = \phi'(u) \neq 0$ .

We shall study the development of a singularity in the classical  $C^1$ -solution  $u$  of (1.1) along the characteristic  $x = x(t, \xi)$  through any point  $\xi \in \mathbb{R}$ , defined to be the unique solution of the initial value problem

$$\frac{dx}{dt} = \phi'(u(x(t, \xi), t)), \quad x(0, \xi) = \xi. \quad (3.5)$$

Assumption (2.1) and the classical theory of ODE guarantee that  $x(t, \xi)$  exists for as long as the  $C^1$ -solution  $u$  of (1.1) exists and has  $u_x(x, t)$  (and hence also  $u_t$ ) bounded. Under the present hypotheses  $x(t, \xi)$  exists for  $0 \leq t < T_1$  for any  $\xi \in \mathbb{R}$ .

Let  $x(t, \xi)$  denote the characteristic curve through  $\xi$  associated with (1.1) which satisfies (3.5). The derivative of  $r$  along this characteristic is

$$\begin{aligned} \frac{dr}{dt} &\equiv r_t + \phi'(u)r_x = z_t + \phi'(u)u_t + \phi'(u)[z_x + \phi'(u)u_x] \\ &= z_t + \phi'(u)[- \phi'(u)u_x - z_x] + \phi'(u)[z_x + \phi'(u)u_x] \\ &= z_t = s_t, \quad 0 \leq t < T_1. \end{aligned}$$

Thus, we may replace (3.2) by the system

$$\begin{aligned} \frac{dr}{dt} &= s_t \\ s_t &= s'(0)\phi(u) + s''(0)\phi(u) \end{aligned} \quad (0 \leq t < T_1), \quad (3.6)$$

together with the initial data  $r(u, s)(x, 0) = \phi(u_0(x))$ ,  $s(u, s)(x, 0) = 0$ , and then by (3.4),  $u = \phi^{-1}(r - s)$ . It is clear that the above calculations are valid for as long as

$u$  is a classical solution of (1.1), i.e., for  $(x,t) \in \mathbb{R} \times [0, T_1]$ . To keep the notation simple it should be understood that when calculating derivatives along a characteristic  $x = x(t, \xi)$ ,  $r = r(x(t, \xi), t) = r(u(x(t, \xi), t), s(x(t, \xi), t))$  and similarly for  $s$ .

To proceed with the proof of Theorem 2.3, let  $v(t, \xi) \equiv x_\xi(t, \xi)$ ,  $0 < t < T_1$ . The function  $v$  measures the variation of two nearby characteristics at time  $t$  with respect to their initial positions and plays a key role in our analysis. When  $v$  is different from zero (1.1) and (3.6) are equivalent. Note that  $v(0, \xi) = 1$  for any  $\xi \in \mathbb{R}$ . We will show that if  $|u_0(\xi)|$  is sufficiently small and  $-u'_0(\xi)$  is sufficiently large, then  $v(t, \xi)$  approaches zero at a finite time  $t_1 < T_1$ , while  $u_\xi(x(t, \xi), t)$  remains finite and bounded away from zero. Observing that

$$u_x(x(t, \xi), t) = \frac{u_\xi(x(t, \xi), t)}{v(t, \xi)} \quad (3.7)$$

we then obtain a contradiction of the assumption that  $u_x$  remains bounded for all  $t \in [0, T_1]$ , and the proof is complete.

Differentiation of (3.5) with respect to  $\xi$  yields

$$\frac{dv}{dt} = \phi''(u(x(t, \xi), t)) u_\xi(x(t, \xi), t), \quad v(0, \xi) = 1, \quad t \in [0, T_1], \quad (3.8)$$

Since  $\phi(u) = r - s$ , we have

$$\phi'(u) u_\xi = r_\xi - s_\xi = r_\xi - s_x x_\xi, \quad (3.9)$$

thus

$$u_\xi = \frac{1}{\phi'(u)} r_\xi - \frac{1}{\phi'(u)} s_x v.$$

From (3.2) and (3.4) the derivative of  $u$  along the characteristic  $x = x(t, \xi)$  is

$$\frac{du}{dt} = -s_x = -s_x,$$

so that

$$u_\xi = \frac{1}{\phi'(u)} r_\xi + \frac{1}{\phi'(u)} \frac{du}{dt} v$$

and (3.8) takes the form

$$\frac{dv}{dt} = \frac{\phi''(u)}{\phi'(u)} r_\xi + \frac{\phi''(u)}{\phi'(u)} \frac{du}{dt} v, \quad v(0, \xi) = 1, \quad t \in [0, T_1].$$

The above equation is an ODE for  $v$  along characteristics having  $[\phi'(u)]^{-1}$  as an integrating factor. Thus

$$\frac{1}{\phi'(u)} v(t, \xi) - \frac{1}{\phi'(u_0(\xi))} = \int_0^t \frac{\phi''(u)}{[\phi'(u)]^2} r_\xi d\tau,$$

or equivalently

$$v(t, \xi) = \frac{\phi'(u(x(t, \xi), t))}{\phi'(u_0(\xi))} \left[ 1 + \phi'(u_0(\xi)) \int_0^t \frac{\phi''(u(x(\tau, \xi), \tau))}{[\phi'(u(x(\tau, \xi), \tau))]^2} r_\xi(x(\tau, \xi), \tau) d\tau \right], \quad (3.10)$$

for  $t \in [0, T_1]$ .

We will now use the following result which provides a bound for  $u$ , independent of  $u_0'(\xi)$ . Its proof is given in Section 4.

Lemma 3.1. Let the assumptions of Theorem (2.3) be satisfied and let  $u$  be a  $C^1$ -smooth solution of (1.1) with  $u, u_x, u_t$  bounded on  $\mathbb{R} \times [0, T_1]$ . Then for any  $\delta > 0$  there exists a number  $\eta = \eta(\delta, T_1) > 0$  such that

$$\sup_{\mathbb{R} \times [0, T_1]} |u(x, t)| < \delta, \quad \text{whenever} \quad \sup_{\mathbb{R}} |u_0(x)| < \eta. \quad (3.11)$$

For a given  $\delta > 0$  we choose  $u_0$  and  $\eta$  in accordance with Lemma 3.1. Since  $\phi'(\cdot)$  and  $\phi''(\cdot)$  are continuous and  $\sup_{\mathbb{R} \times [0, T_1]} |u(x, t)| < \delta$ , assumptions (2.1), (2.2), and (2.3) imply that there exists positive constants  $\alpha_i, i = 1, \dots, 4$  such that

$$\begin{aligned} \alpha_1 &< \phi'(u(x(t, \xi), t)) < \alpha_2, \quad \phi''(u(x(t, \xi), t)) > \alpha_3 \\ \alpha_4 &< \frac{\phi'(u(x(t, \xi), \tau))}{\phi'(u(x(t, \xi), \tau))} < \beta \end{aligned} \quad (3.12)$$

for  $0 < \tau < t < T_1$ , where  $\beta$  is the a priori constant in (2.3). We note that the constants  $\alpha_i$  depend on  $\delta$  but not on  $u_0'(\xi)$ .

To proceed with the proof we shall also need to estimate  $r_\xi$  in (3.10), as well as  $r_\xi - s_\xi$  in (3.9). For this purpose note from (3.4), (3.9) that

$$r_\xi(x(0, \xi), 0) = \phi'(u_0(\xi))u_0'(\xi) \quad (\xi \in \mathbb{R}).$$

Let  $C(\xi)$  and  $C^*$  be defined by

$$C(\xi) = -\phi'(u_0(\xi))u_0'(\xi) \quad (3.13)$$

$$C^* = \sup_{\xi \in R} |C(\xi)|.$$

We note that  $C(\xi)$  is positive whenever  $u_0'(\xi)$  is negative. We will now use the following auxiliary result; its proof is given in Section 4.

Lemma 3.2. Let the assumptions of Lemma 3.1 be satisfied. Select the datum  $u_0$  such that  $u_0'(\xi) < 0$ , and there is a point  $\xi_0$  such that  $C(\xi_0) = C^*$ . Then there exists  $0 < T_2 < T_1$ , independent of  $C^*$  (hence of  $u_0'(\xi_0)$ ), such that

$$-\frac{7}{4}C^* \leq r_\xi(x(t, \xi_0), t) - s_\xi(x(t, \xi_0), t) \leq -\frac{C^*}{4}, \quad -\frac{3C^*}{2} \leq r_\xi(x(t, \xi_0), t) \leq -\frac{C^*}{4} \quad (3.14)$$

for  $0 < t < T_2$ .

To complete the proof use equation (3.10) and the inequalities (3.12), as well as the inequality for  $r_\xi(x(t, \xi_0), t)$  in (3.14), to obtain the estimate

$$v(t, \xi) \leq \frac{\phi'(u(x(t, \xi_0)))}{\phi'(u_0(\xi_0))} \left[ 1 - \frac{C^*}{4} \frac{a_3 a_1}{a_2^2} t \right] \quad (3.15)$$

for  $0 < t < T_2$ . By (3.12)  $\phi'(u(x(t, \xi_0)))$  is finite and bounded away from zero for

$0 < t < T_2$ . Thus the right hand side of (3.15) becomes zero at time  $t_1^* = \frac{4a_2^2}{a_1 a_3 C^*}$ .

Since  $T_2$  is independent of  $u_0'(\xi_0)$ , we now choose  $C^*$  (i.e.  $-u_0'(\xi_0) > 0$ ) so large (cf. (3.13)) that  $t_1^* < T_2$ , while keeping  $u_0(\xi_0)$  fixed and  $|u_0(\xi_0)| < \eta$ . Finally, by (3.9), (3.12) and the first inequality in (3.14)  $u_\xi(x(t, \xi_0), t)$  remains finite and bounded away from zero on  $0 < t < T_2$ . Equation (3.7) then provides the desired contradiction (i.e.  $u_x$  becomes negatively infinite at some time  $t_1 < t_1^*$ ); this completes the proof of Theorem 3.2.

#### 4. PROOFS OF LEMMA 3.1 AND 3.2.

a. Proof of Lemma 3.1. It follows from (2.3), (2.4), (3.4) and (3.6) that

$$\begin{aligned} \frac{dx}{dt}(x(t, \xi), t) &\leq \beta |a'(0)| [|x(x(t, \xi), t)| + |s(x(t, \xi), t)|] \\ &+ \beta \int_0^t a''(t - \tau) [|x(x(t, \xi), \tau)| + |s(x(t, \xi), \tau)|] d\tau \end{aligned} \quad (4.2)$$

$$s_t(x, t) \leq \beta |a'(0)| [|x(x, t)| + |s(x, t)|] + \beta \int_0^t a''(t - \tau) [|x(x, \tau)| + |s(x, \tau)|] d\tau$$

for  $0 < t < T_1$ . Let  $R(t)$  and  $S(t)$  be defined by

$$R(t) = \sup_{x \in \mathbb{R}} |x(x, t)|, \quad S(t) = \sup_{x \in \mathbb{R}} |s(x, t)|. \quad (4.3)$$

Integrating the inequalities (4.2), taking the supremum on the right hand side and using the definitions (4.3), we obtain

$$\begin{aligned} |x(x(t, \xi), t)| &\leq \sup_{\xi \in \mathbb{R}} |x_0(\xi)| + \beta |a'(0)| \int_0^t [R(\tau) + S(\tau)] d\tau + \\ &+ \beta \int_0^t \int_0^\eta a''(\eta - \tau) [R(\tau) + S(\tau)] d\tau d\eta \end{aligned} \quad (4.4)$$

$$|s(x, t)| \leq \beta |a'(0)| \int_0^t [R(\tau) + S(\tau)] d\tau + \beta \int_0^t \int_0^\eta a''(\eta - \tau) [R(\tau) + S(\tau)] d\tau d\eta$$

for  $0 < t < T_1$ , where  $x_0(\xi) = x(x(0, \xi), 0) = \phi(u_0(\xi), s(x(0, \xi), 0)) = 0$ . We note that the right hand side of (4.4) is independent of  $x$  and  $\xi$ . Moreover, from the smoothness of  $u$ ,  $u_t$ , and  $u_x$ , assumption (2.1), and the continuous dependence of solutions of equation (3.5) on the initial data, it follows readily that for each fixed  $t$ ,  $t < T_1$ , there exists  $\xi \in \mathbb{R}$  and  $x(t, \xi)$  such  $x(x(t, \xi), t) = R(t)$  and  $s(x(t, \xi), t) = S(t)$  hold. Therefore, we can replace the left hand sides of (4.4) by  $R(t)$  and  $S(t)$  respectively. Interchanging the order of integration in the double integrals in (4.4) yields



$$R(t) \leq \sup_R |x_0(\xi)| + 2\beta |a'(0)| \int_0^t [R(\tau) + S(\tau)] d\tau + \beta \int_0^t a'(t-\tau) [R(\tau) + S(\tau)] d\tau \quad (4.5)$$

$$S(t) \leq 2\beta |a'(0)| \int_0^t [R(\tau) + S(\tau)] d\tau + \beta \int_0^t a'(t-\tau) [R(\tau) + S(\tau)] d\tau$$

for  $0 \leq t \leq T_1$ . We add the two inequalities in (4.5) to obtain

$$R(t) + S(t) \leq \sup_R |x_0(\xi)| + \int_0^t [4\beta |a'(0)| + 2\beta a'(t-\tau)] [R(\tau) + S(\tau)] d\tau \quad (4.6)$$

for  $0 \leq t \leq T_1$ . Let  $H(t) = \max_{0 \leq \tau \leq t} \{4\beta |a'(0)| + 2\beta a'(t-\tau)\}$ , which is a nonnegative function by (2.4). Thus

$$R(t) + S(t) \leq \sup_R |x_0(\xi)| + H(t) \int_0^t [R(\tau) + S(\tau)] d\tau, \quad 0 \leq t \leq T_1, \quad (4.7)$$

and the Gronwall inequality yields the estimate

$$R(t) + S(t) \leq \sup_R |x_0(\xi)| f(t), \quad 0 \leq t \leq T_1, \quad (4.8)$$

where the positive function  $f(\cdot)$  is defined by

$$f(t) = 1 + H(t) \int_0^t (\exp \int_s^t H(\tau) d\tau) ds, \quad 0 \leq t \leq T_1. \quad (4.9)$$

Since  $x_0(\xi) = \phi(u_0(\xi))$ , inequality (4.8), equations (3.4), and the monotonicity of  $f$  imply that

$$|\phi(u(x,t))| \leq \sup_R |\phi(u_0(\xi))| f(T_1) \quad (4.10)$$

for  $(x,t) \in R \times [0, T_1]$ . Recalling that  $\phi'(\cdot) > 0$ , we observe that (4.10) is equivalent to

$$|u(x,t)| \leq \phi^{-1}(\sup_R |\phi(u_0(\xi))| f(T_1)) \quad (4.11)$$

for  $(x, t) \in \mathbb{R}^n \times [0, T_1]$ . The proof of the lemma now follows from the continuity of  $\phi$  and  $\phi^{-1}$  and the fact that  $\phi(0) = 0$ .

b. Proof of Lemma 3.2. We write the system (3.6) in the equivalent form

$$\frac{dx}{dt} = a'(0)\phi(u(x(t, \xi), t)) + \int_0^t a''(t-\tau)\phi(u(x(t, \xi), \tau))d\tau, \quad (4.12)$$

$$s(x, t) = \int_0^t a'(t-\tau)\phi(u(x, \tau))d\tau, \quad t \in [0, T_1].$$

Integrating (4.12) with respect to  $t$ , differentiating the outcome with respect to  $\xi$  and using (3.9), we obtain

$$\begin{aligned} x_\xi(x(t, \xi), t) = & -C(\xi) + a'(0) \int_0^t \frac{\phi'(u(x(\tau, \xi), \tau))}{\phi'(u(x(\tau, \xi), \tau))} [x_\xi(x(\tau, \xi), \tau) - s_\xi(x(\tau, \xi), \tau)] d\tau \\ & + \int_0^t \int_0^\tau a''(\tau-\eta) \frac{\phi'(u(x(\tau, \xi), \eta))}{\phi'(u(x(\tau, \xi), \eta))} [x_\xi(x(\tau, \xi), \eta) - s_\xi(x(\tau, \xi), \eta)] d\eta d\tau \end{aligned} \quad (4.13)$$

$$s_\xi(x, t) = \int_0^t a'(t-\tau) \frac{\phi'(u(x, \tau))}{\phi'(u(x, \tau))} [x_\xi(x, \tau) - s_\xi(x, \tau)] d\tau.$$

Define  $\rho$  and  $\sigma$  by

$$\rho(t) = \sup_{x \in \mathbb{R}^n} |x_\xi(x, t)|, \quad \sigma(t) = \sup_{x \in \mathbb{R}^n} |s_\xi(x, t)|. \quad (4.14)$$

Next, we take absolute values of both sides in (4.13), use the definitions (4.14) and inequalities (3.12) to obtain the inequalities

$$\begin{aligned} |x_\xi(x(t, \xi), t)| \leq & C^* + \beta |a'(0)| \int_0^t [\rho(\tau) + \sigma(\tau)] d\tau + \\ & + \beta \int_0^t \int_0^\tau a''(\tau-\eta) [\rho(\eta) + \sigma(\eta)] d\eta d\tau \end{aligned} \quad (4.15)$$

$$|s_\xi(x, t)| \leq \beta \int_0^t |a'(t-\tau)| [\rho(\tau) + \sigma(\tau)] d\tau,$$

where  $C^*$  is defined in (3.13). Let  $I(t) = \rho(t) + \sigma(t)$ . As in the proof of Lemma 3.1, we can replace the left hand sides of (4.15) by  $\rho(t)$  and  $\sigma(t)$ . After simplifying the first inequality in (4.15) by interchanging the order in the double integral and adding the two inequalities (4.15), we obtain

$$I(t) \leq C^* + 2\delta \int_0^t [|a'(0)| + |a'(t-\tau)|] I(\tau) d\tau, \quad 0 \leq t \leq T_1. \quad (4.16)$$

Noting that  $\max_{0 \leq \tau \leq t} [|a'(0)| + |a'(t-\tau)|] = 2|a'(0)|$  (cf. (2.4)), (4.16) becomes

$$I(t) \leq C^* + 4\delta |a'(0)| \int_0^t I(\tau) d\tau, \quad 0 \leq t \leq T_1, \quad (4.17)$$

which, by the Gronwall inequality, implies that

$$I(t) \leq C^* \exp(4\delta |a'(0)| t), \quad 0 \leq t \leq T_1. \quad (4.18)$$

We now choose  $T_2^* < T_1$  small enough so that

$$I(t) \leq \frac{3C^*}{2}, \quad t \in [0, T_2^*]. \quad (4.19)$$

Note that  $T_2^*$  depends only on  $u_0(\xi)$  and  $a(\cdot)$ , and  $T_2^*$  is independent  $C^*$ .

Inequalities (4.15) and (4.19) combine to yield

$$|u_\xi(x, t)| \leq \frac{3C^*}{2} \delta \int_0^t |a'(\eta)| d\eta, \quad 0 \leq t \leq T_2^*. \quad (4.20)$$

We further restrict  $T_2^*$  so that

$$|u_\xi(x, t)| \leq \frac{C^*}{4}, \quad 0 \leq t \leq T_2^*, \quad x \in \mathbb{R}. \quad (4.21)$$

We observe that up to this point the sign of  $u_0(\xi)$  plays no role and the estimates (4.19), (4.21) hold for any  $\xi \in \mathbb{R}$ .

We next turn to estimating  $r_\xi(x(t, \xi), t)$ ; the estimate  $|r_\xi(x(t, \xi), t)| \leq \frac{3}{2} C^*$  for  $0 \leq t \leq T_2^*$ , which follows trivially from (4.19), is too crude to establish Lemma 3.2. We

now select the datum  $u_0$  and a point  $\xi_0$  as specified in the statement of Lemma 3.2. The goal is to obtain a negative upper bound for  $r_\xi(x(t, \xi_0), t)$ ; this is obtained from the first equation in (4.13) as follows. Using (3.12) and estimating the two integrals on the right hand side of (4.13) as in (4.15), (4.16), and then using (4.19), we obtain the estimate

$$\begin{aligned} |a'(0)| \int_0^t \frac{\phi'(u(x(\tau, \xi), \tau))}{\phi'(u(x(\tau, \xi), \tau))} |r_\xi(x(\tau, \xi), \tau) - s_\xi(x(\tau, \xi), \tau)| d\tau \\ + \int_0^t \int_0^\tau a''(\tau - \eta) \frac{\phi'(u(x(\tau, \xi), \eta))}{\phi'(u(x(\tau, \xi), \eta))} |r_\xi(x(\tau, \xi), \eta) - s_\xi(x(\tau, \xi), \eta)| d\eta d\tau \quad (4.22) \\ < 3C^* \delta |a'(0)| t, \end{aligned}$$

for  $0 < t < T_2^*$ . Putting  $\xi = \xi_0$  in (4.13) and then using (4.22) gives

$$r_\xi(x(t, \xi_0), t) < -C^* + 3C^* \delta |a'(0)| t, \quad 0 < t < T_2^*, \quad (4.23)$$

where  $T_2^*$  is independent of  $C^*$ . Then choosing  $0 < T_2 < T_2^*$  small enough and independently of  $C^*$  we obtain

$$r_\xi(x(t, \xi_0), t) < -\frac{C^*}{4}, \quad 0 < t < T_2. \quad (4.24)$$

This, together with the crude lower bound  $(-\frac{3}{2} C^*)$  already mentioned proves the second set of desired inequalities in (3.14). These combined with (4.21) (which of course holds  $0 < t < T_2 < T_2^*$ ) yield the first set of inequalities in (3.14), and the proof of Lemma 3.2 is complete.

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ABSTRACT (cont.)

is a given memory kernel, subscripts denote partial derivatives,  $' = d/dt$  and  $*$  denotes the convolution on  $[0, t]$ . Under physically reasonable assumptions concerning the functions  $\phi, \psi$  and  $a$  it is shown that a smooth solution  $u$  develops a singularity in finite time, whenever the smooth datum  $u_0$  becomes "sufficiently large" in a precise sense.